# CS 58500 – Theoretical Computer Science Toolkit

## Lecture 7 (02/10)

## Asymptotic Convex Geometry

https://ruizhezhang.com/course_spring_2026.html

# Today's Lecture

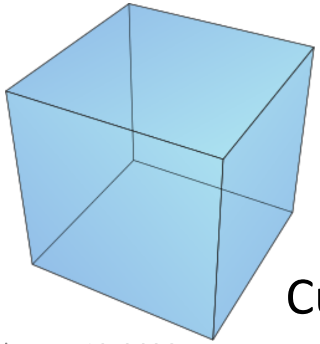
- Introduction to Asymptotic Convex Geometry
  - Geometric Intuition in High Dimensions
  - Key Questions in ACG (A Bird's-Eye View)
- Why TCS Cares About ACG

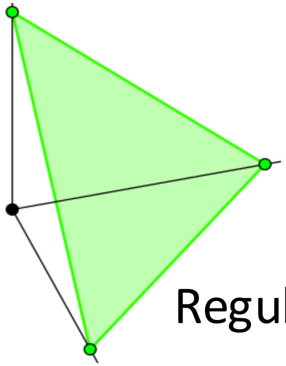
# Introduction to Asymptotic Convex Geometry

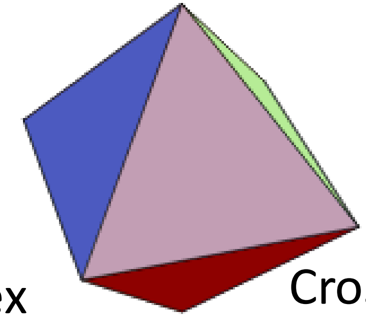
- The study of geometric and linear properties of finite dimensional normed spaces or convex bodies, focusing on what happens as the dimension $n \to \infty$
- A theory that lies between the classical Convex Geometry (Brunn-Minkowski theory; fixed dimension) and the classical Functional Analysis ($\infty$-dimension)
- Keith Ball quoted in his wonderful survey *An Elementary Introduction to Modern Convex Geometry* that:

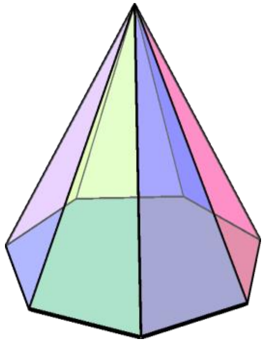
"All convex bodies behave a bit like Euclidean balls."

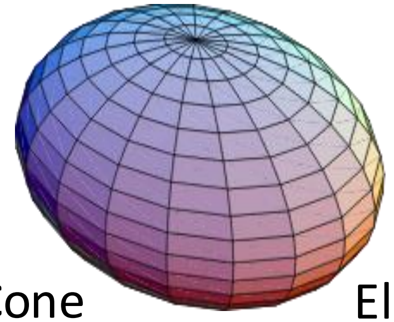
- This claim is "almost true" if one adds a few extra shapes:


Cube


Regular simplex


Cross-polytope

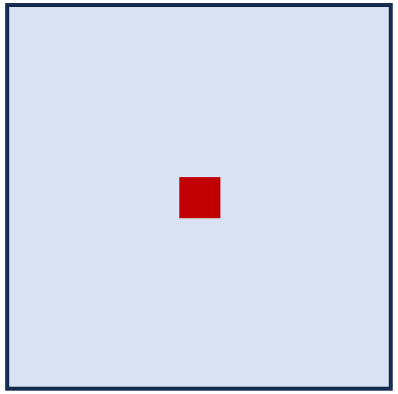

Cone

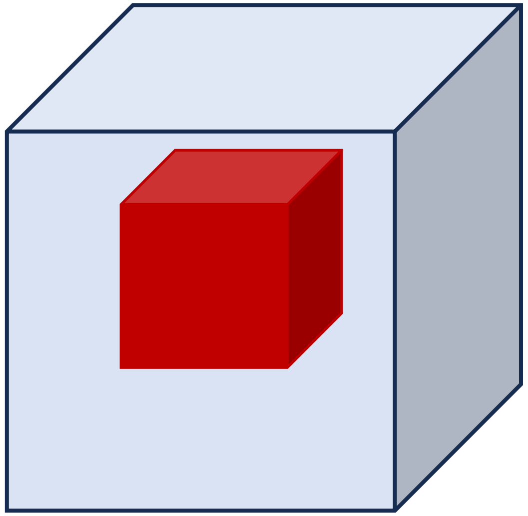

Ellipsoid

# Geometric Intuition in High Dimensions

The vast majority of volume lies near the boundary of a convex body

- In $\mathbb{R}^2$, to get 1% volume of the square $[-1,1]^2$, we can take $[-0.1,0.1]^2$
- In $\mathbb{R}^3$, to get 1% volume of the cube $[-1,1]^3$, we can take $[-0.43,0.43]^3$
- In $\mathbb{R}^{100}$, to get 1% volume of the cube $[-1,1]^{100}$, we need to take $[-0.$





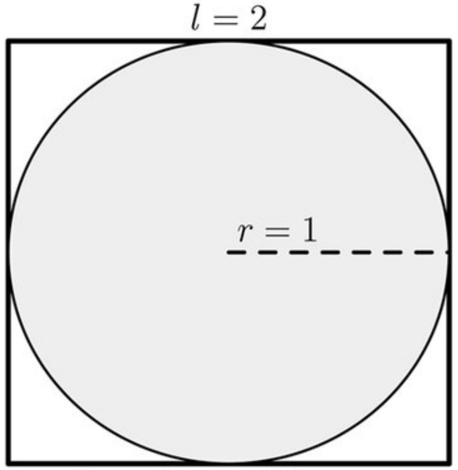
# Geometric Intuition in High Dimensions

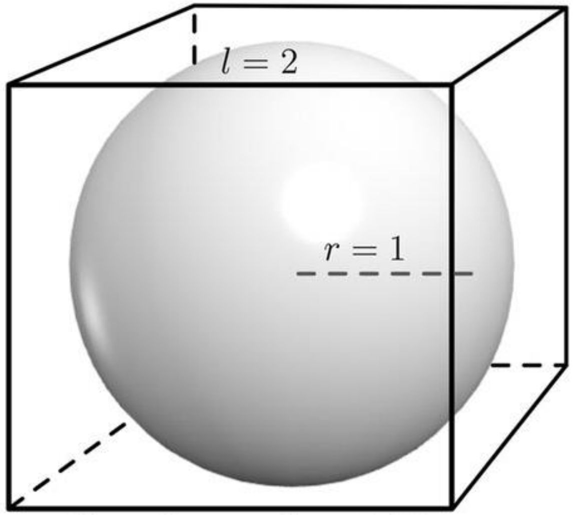
## Cubes are very different from balls in high dimensions

- In $\mathbb{R}^n$, a cube with volume 1 has side-length 1
- The volume of the Euclidean ball $B_2^n$ is

$$\mathrm{Vol}_n(B_2^n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \sim \left(\frac{2\pi e}{n}\right)^{n/2}$$

- To get volume 1, the radius $r \sim \sqrt{n}$
- That is, balls in high dimensions are much smaller than cubes
- Consider a sphere maximally inscribed in a cube (i.e., $\mathbb{S}^{n-1}$ inside of $[-1,1]^n$). Then very little of the sphere is near the cube's boundary

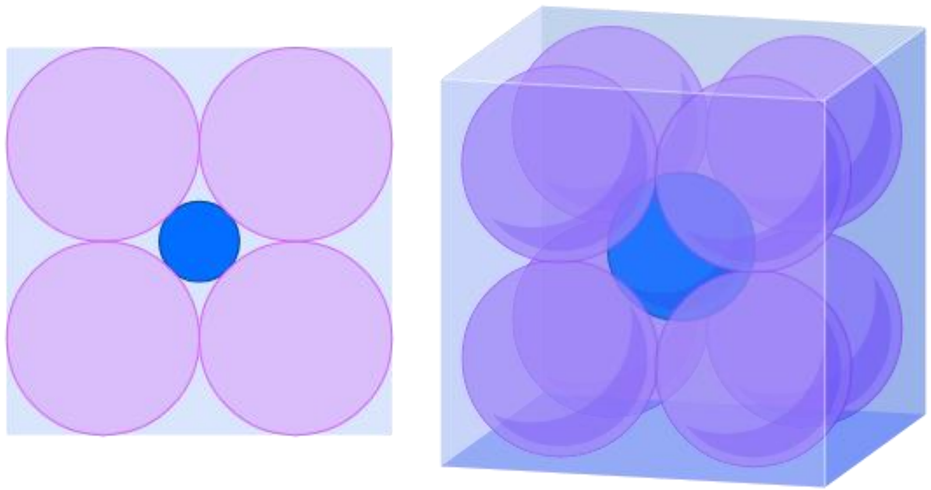


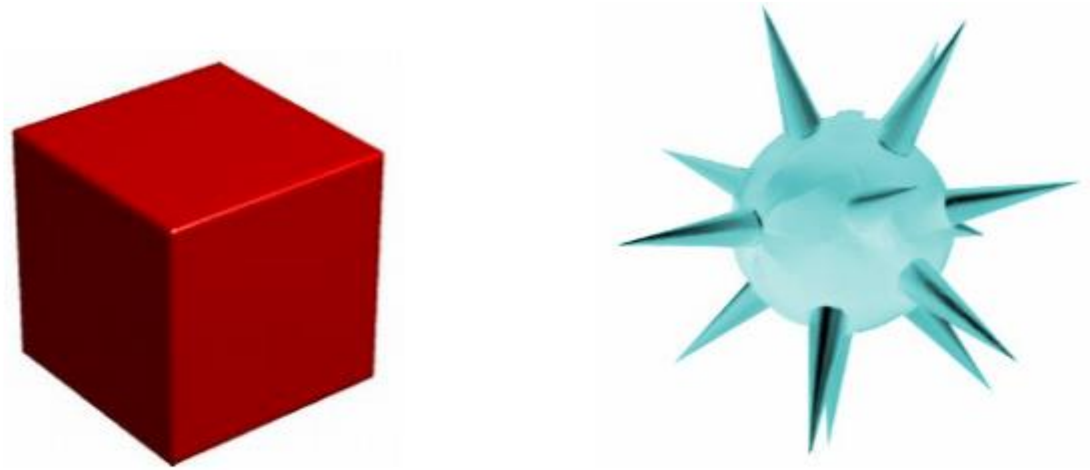



# Geometric Intuition in High Dimensions

## Cubes are very different from balls in high dimensions

- Consider a sphere maximally inscribed in a cube (i.e., $\mathbb{S}^{n-1}$ inside of $[-1,1]^n$). Then very little of the sphere is near the cube's boundary
- Suppose, we place $2^n$ spheres of radius 1 inside $[-2,2]^n$, and then put another one in the center tangent to them all. It has radius $\sqrt{n} - 1$
- When $n \geq 10$, the central sphere protrudes beyond the sides of the cube

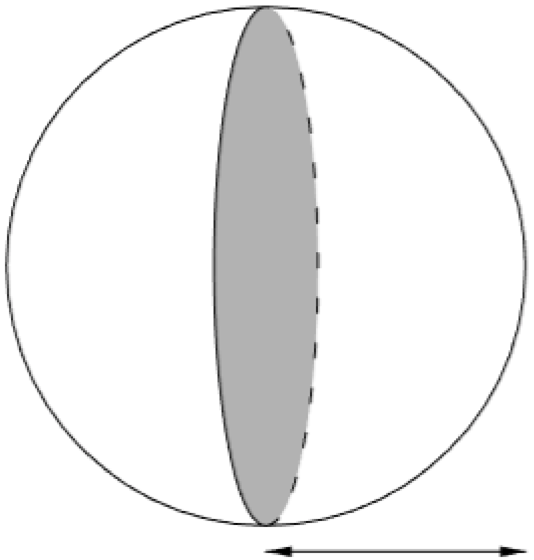




High dimensional cubes are spiky

# Geometric Intuition in High Dimensions

The volume of a high-dimensional ball is concentrated around a central slice




- For an $r$-radius ball with volume 1, consider $\mathrm{Vol}_{n-1}(rB_2^n \cap \{x_1 = 0\})$
- $r = \mathrm{Vol}_n(B_2^n)^{-1/n} \approx \sqrt{n/(2\pi e)}$

$$\begin{aligned} \mathrm{Vol}_{n-1}(rB_2^n \cap \{x_1 = 0\}) &= \mathrm{Vol}_{n-1}(rB_2^{n-1}) \\ &= \mathrm{Vol}_{n-1}(B_2^{n-1})\mathrm{Vol}_n(B_2^n)^{-(n-1)/n} \\ &\approx \sqrt{e} \end{aligned}$$

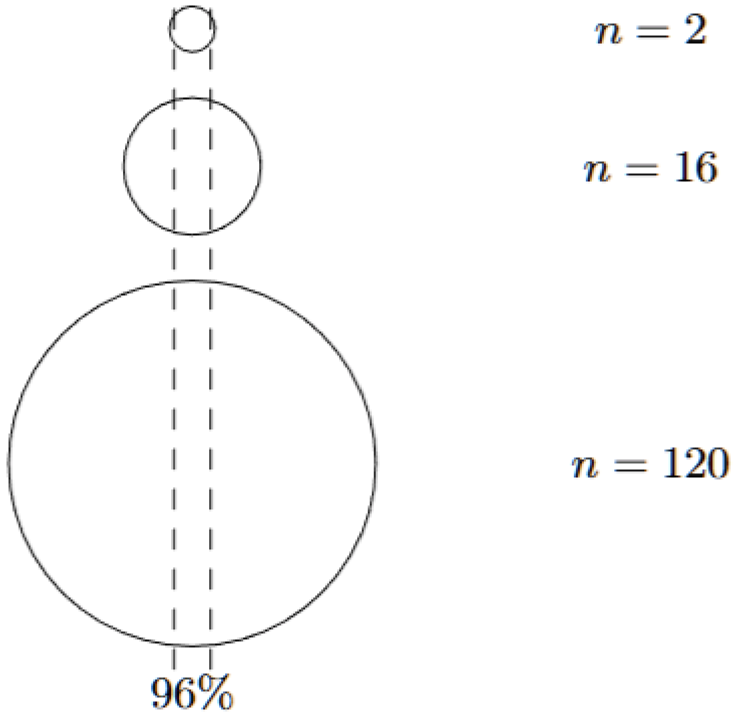
- Let $v(t) \coloneqq \mathrm{Vol}_{n-1}(rB_2^n \cap \{x_1 = t\})$
- $v(t) = v(0)\left(\frac{\sqrt{r^2-t^2}}{r}\right)^{n-1} \approx \sqrt{e}\left(1 - \frac{2\pi e t^2}{n}\right)^{\frac{n-1}{2}} \approx \sqrt{e}\exp(-\pi e x^2)$
- The volume distribution is a Gaussian with variance $1/(2\pi e)$ (dimension-free!)

# Geometric Intuition in High Dimensions

The volume of a high-dimensional ball is concentrated around a central slice

- Let $v(t) \coloneqq \mathrm{Vol}_{n-1}(rB_2^n \cap \{x_1 = t\})$
- $v(t) = v(0)\left(\frac{\sqrt{r^2-t^2}}{r}\right)^{n-1} \approx \sqrt{e}\left(1 - \frac{2\pi e t^2}{n}\right)^{\frac{n-1}{2}} \approx \sqrt{e}\,\exp(-\pi e x^2)$
- The volume distribution is a Gaussian with variance $1/(2\pi e)$
- About 96% of the volume lies in the slab

$$\left\{\boldsymbol{x} \in rB_2^n : -\frac{1}{2} \le x_1 \le \frac{1}{2}\right\}$$



# Key Questions in ACG (A Bird’s-Eye View)

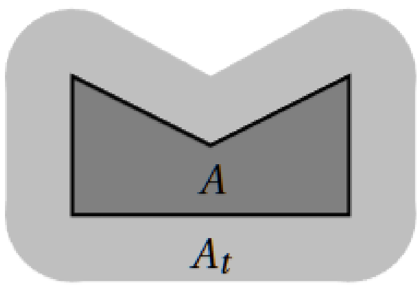
- Isoperimetric and Concentration of Measure
- Covering the Convex Bodies
- Euclidean Structure Inside Arbitrary Norms
- Putting Convex Bodies in the Right Positions
- Distribution of Mass in Convex Bodies

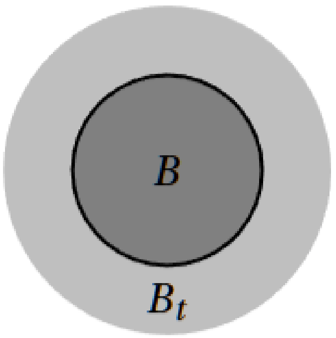
# Isoperimetric and Concentration of Measure

- It was known to the Greeks that among all closed plane curves with a fixed perimeter, the circle encloses the maximum possible area
- More generally and equivalently, among bodies in $\mathbb{R}^n$ with identical volumes, the Euclidean ball $B_2^n$ minimizes the surface area
- Define the distance of a point $\boldsymbol{x}$ to a set $A$ as $d(\boldsymbol{x}, A) \coloneqq \inf\{\|\boldsymbol{x} - \boldsymbol{y}\|_2 : \boldsymbol{y} \in A\}$
- Let $A_t \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n : d(\boldsymbol{x}, A) \leq t\}$ be the $t$-neighborhood of $A$

**Isoperimetric inequality in Euclidean space (**Brunn-Minkowski**).**

Let $A \subseteq \mathbb{R}^n$ be a compact set and let $B \coloneqq rB_2^n$ be the Euclidean ball so that $\mathrm{Vol}_n(A) = \mathrm{Vol}_n(B)$. Then $\mathrm{Vol}_n(A_t) \geq \mathrm{Vol}_n(B_t)$ for any $t \geq 0$

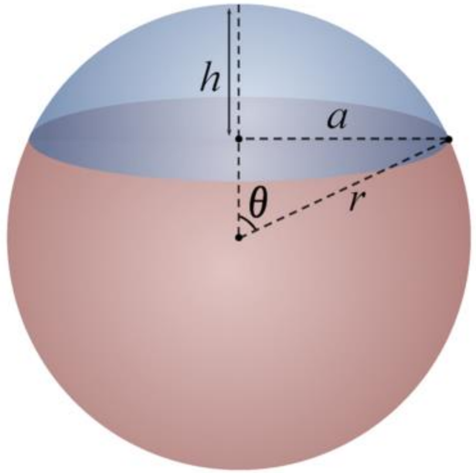



Surface area:

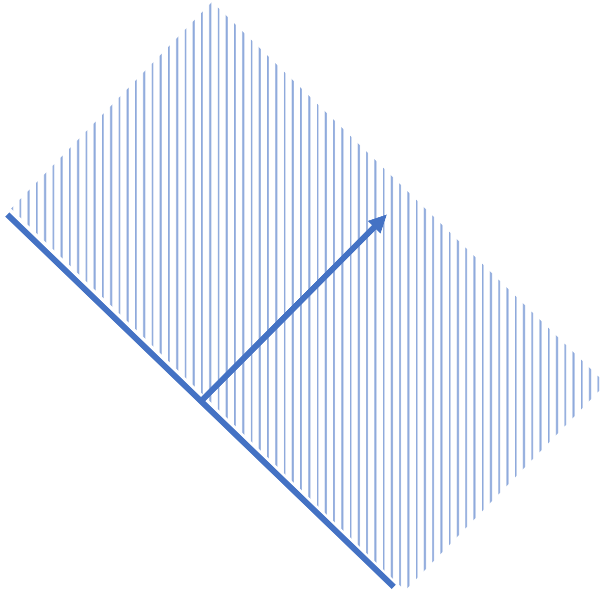
$$\mathrm{Vol}_{n-1}(\partial A) \coloneqq \lim_{\epsilon \to 0} \frac{\mathrm{Vol}_n(A + \epsilon B_2^n) - \mathrm{Vol}_n(A)}{\epsilon}$$

# Isoperimetric and Concentration of Measure




**Isoperimetric inequality on sphere (**Lévy, Schmidt**).**

Let $\mathbb{S}^{n-1} := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2 = 1\}$ and $\mu$ be the uniform measure over it. For any $A \subseteq \mathbb{S}^{n-1}$, let $B$ be a spherical cap such that $\mu(A) = \mu(B)$. Then $\mu(A_t) \geq \mu(B_t)$



**Isoperimetric inequality in Gaussian space (**Borell, Sudakov-Tsirelson**).**

Let $\gamma_n$ be the standard Gaussian distribution in $\mathbb{R}^n$. For any measurable set $A \subseteq \mathbb{R}^n$, let $H = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{\theta}, \boldsymbol{x} \rangle \leq \lambda\}$ be a halfspace such that $\gamma_n(A) = \gamma_n(H)$. Then $\gamma_n(A_t) \geq \gamma_n(H_t)$

# Isoperimetric and Concentration of Measure

- Vitali Milman recognized the importance of the concentration of measure phenomenon, which he heavily promoted in the 1970's

Probability

Concentration of Lipschitz functions

For a 1-Lipschitz $f: \mathbb{S}^{n-1} \to \mathbb{R}$,

$$\Pr_{\boldsymbol{x}\sim\mu}[|f(\boldsymbol{x}) - \mathrm{med}(f)| \geq t] \leq 2\exp(-nt^2/4)$$

For a 1-Lipschitz $f: \mathbb{R}^n \to \mathbb{R}$,

$$\Pr_{\boldsymbol{x}\sim\gamma_n}[|f(\boldsymbol{x}) - \mathrm{med}(f)| \geq t] \leq 2\exp(-t^2/2)$$

Geometry

Isoperimetric inequalities

For any $A \subseteq \mathbb{S}^{n-1}$ with $\mu(A) \geq 1/2$

$$\mu(A_t) \geq 1 - 2\exp(-nt^2/4)$$

For any $A \subseteq \mathbb{R}^n$ with $\mu(A) \geq 1/2$

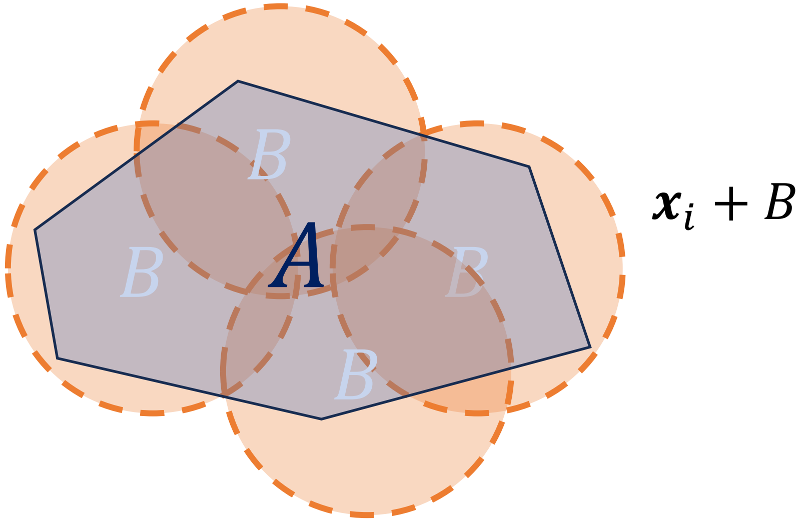
$$\gamma_n(A_t) \geq 1 - \exp(-t^2/2)$$

$$\boxed{A := \{\boldsymbol{x} : f(\boldsymbol{x}) \leq \mathrm{med}(f)\}}$$

# Covering the Convex Bodies

For two convex bodies $A, B \subseteq \mathbb{R}^n$, we define the covering number $N(A,B)$ as the minimum number of translates of $B$ necessary to cover $A$

$$N(A,B) \coloneqq \min\left\{N \geq 0 : \exists \boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^d \text{ s.t. } A \subseteq \bigcup_{i=1}^{N} (\boldsymbol{x}_i + B)\right\}$$




- Trivial lower bound: $N(A,B) \geq \frac{\mathrm{Vol}_n(A)}{\mathrm{Vol}_n(B)}$
- Affine invariance: for any invertible linear transform $T: \mathbb{R}^n \to \mathbb{R}^n$,
$$N(A,B) = N\big(T(A), T(B)\big)$$
- Highly non-trivial bounds (Milman-Pajor + Rogers-Shephard):

$$4^{-n} \frac{\mathrm{Vol}_n(A-B)}{\mathrm{Vol}_n(B)} \leq N(A,B) \leq 4^n \frac{\mathrm{Vol}_n(A-B)}{\mathrm{Vol}_n(B)}$$

# Covering the Convex Bodies

## Connection to the supremum of a Gaussian process

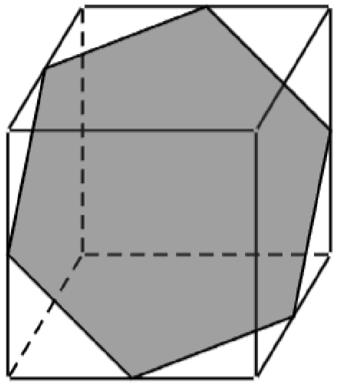
- Let $\mathrm{T}$ be an abstract set. A random process on $\mathrm{T}$ is a family $(X_t : t \in \mathrm{T})$ of random variables
- $(X_t : t \in \mathrm{T})$ is called a centered Gaussian process if the family $(X_t : t \in \mathrm{S})$ is jointly centered Gaussian for every finite subset $\mathrm{S} \subseteq \mathrm{T}$, i.e., for any $a_t \in \mathbb{R}$, $\sum_{t\in\mathrm{S}} a_t X_t$ is a mean-zero Gaussian random variable
- What is $\mathbb{E}[\sup_{t\in\mathrm{T}} X_t]$?

$$\sup_{\epsilon>0} \epsilon\sqrt{\log N(T, \epsilon B_2^n)} \lesssim \mathbb{E}[\sup_{t\in\mathrm{T}} X_t] \lesssim \int_0^\infty \sqrt{\log N(T, \epsilon B_2^n)}\, \mathrm{d}\epsilon$$

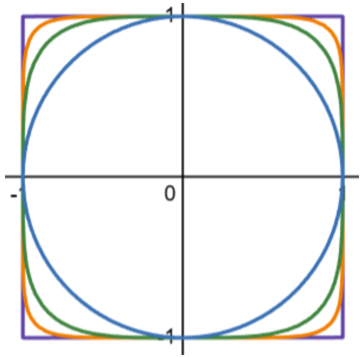
(Sudakov) (Dudley)

# Euclidean Structure Inside Arbitrary Norms

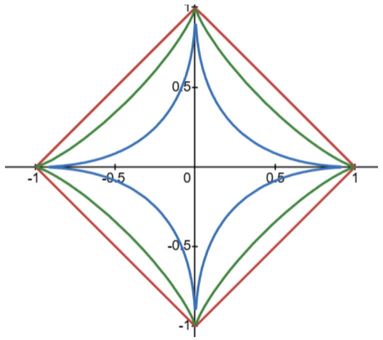
Informal version of **Dvoretzky's theorem**: For any normed space $(\mathbb{R}^n, \|\cdot\|)$ and $k \lesssim \log n$, most $k$-dimensional subspaces look very similar to $k$-dimensional Euclidean space $\left(\mathbb{R}^k, \|\cdot\|_2\right)$



- A norm $\|\cdot\|$ is a map $\mathbb{R}^n \to \mathbb{R}$ satisfying
  - Subadditivity: $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$
  - Homogenity: $\|\lambda \boldsymbol{x}\| \leq |\lambda| \cdot \|\boldsymbol{x}\| \;\; \forall \lambda \in \mathbb{R}$
  - Point-separation: $\|\boldsymbol{x}\| = 0 \implies \boldsymbol{x} = \boldsymbol{0}$
- For a symmetric convex body $K$ (i.e., $K = -K$), the Minkowski norm is defined as
$$\|\boldsymbol{x}\|_K \coloneqq \min\{\lambda \geq 0 : \lambda \boldsymbol{x} \in K\}$$
- Every norm is equivalent to a Minkowski norm with $K \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| = 1\}$




$B_p^2$ $(p > 1)$



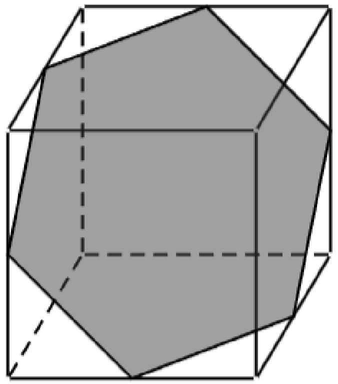

$B_p^2$ $(p \leq 1)$

# Euclidean Structure Inside Arbitrary Norms

**Dvoretzky's theorem (**Dvoretzky, Milman**).**

Let $K \subseteq \mathbb{R}^n$ be a convex symmetric body with $B_2^n \subseteq K$. Let $V \subseteq \mathbb{R}^n$ be a uniformly random subspace of dimension $k \lesssim \frac{\epsilon^2}{\log(1/\epsilon)} \log n$. Then, with very high probability, $K \cap V$ is $(1+\epsilon)$-spherical; that is,

$$\frac{1}{1+\epsilon}\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_K \leq (1+\epsilon)\|\boldsymbol{x}\|_2 \quad \forall \boldsymbol{x} \in V$$



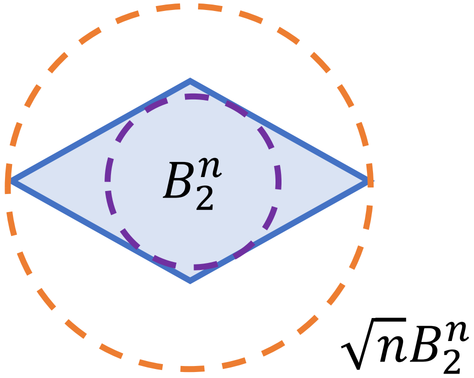
- Universality in high dimensions: If you reduce the dimension sufficiently, what typically happens is that all of the original structure is lost and all you see is this canonical nice (or boring) space

# Putting Convex Bodies in the Right Positions

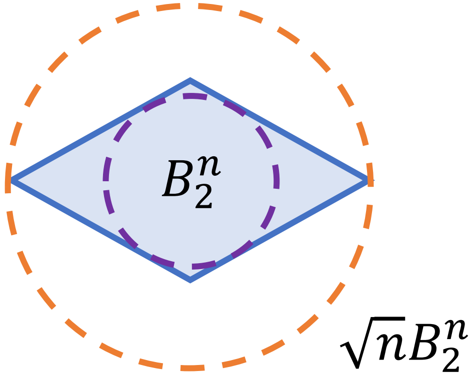
Many results in asymptotic convex geometry are only correct if you put the convex body in the "right" position, i.e., apply a linear transform $T: \mathbb{R}^n \to \mathbb{R}^n$ to the convex body $K$ so that $T(K)$ satisfies the desired properties


John position

- $B_2^n$ is an ellipsoid of maximum volume contained in $K$
- **John's theorem:** Let $K$ be a centrally symmetric convex body in John position. Then $B_2^n \subseteq K \subseteq \sqrt{n} B_2^n$



# Putting Convex Bodies in the Right Positions




John position

- $B_2^n$ is an ellipsoid of maximum volume contained in $K$
- **John's theorem:** Let $K$ be a centrally symmetric convex body in John position. Then $B_2^n \subseteq K \subseteq \sqrt{n}B_2^n$
- **A slightly stronger version (**John, Ball**):** Let $K$ be a centrally symmetric convex body. Then $K$ contains a unique ellipsoid of maximal volume (John ellipsoid).
  Moreover, this largest ellipsoid is $B_2^n$ if and only if the following conditions hold:
  - $B_2^n \subseteq K$
  - There exist $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in \partial K \cap B_2^n$ and $c_1, \dots, c_m > 0$ such that

$$\sum_{i=1}^{m} c_i \boldsymbol{x}_i = \boldsymbol{0}, \qquad \sum_{i=1}^{m} c_i \boldsymbol{x}_i \boldsymbol{x}_i^\top = \boldsymbol{I}_n$$

# Distribution of Mass in Convex Bodies

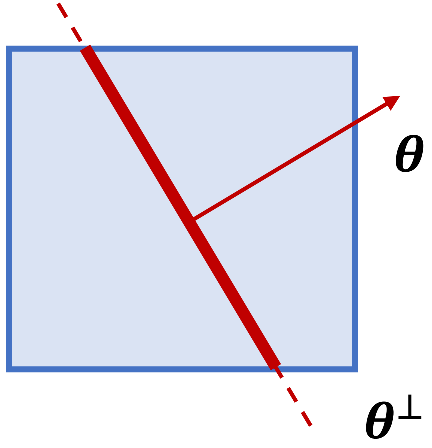
Isotropic position

- A convex body $K$ is in isotropic position if the following conditions are satisfied:
  - $\mathrm{Vol}_n(K) = 1$
  - $\mathbb{E}_{\boldsymbol{x}\sim K}[\boldsymbol{x}] = 0$ ($K$ is centered)
  - $\mathbb{E}_{\boldsymbol{x}\sim K}[\boldsymbol{x}\boldsymbol{x}^\top] = L_k^2 \cdot \boldsymbol{I}_n$, where $L_K \geq 0$ is called the isotropic constant of $K$
- For any centered convex body $K$, there is a linear map $T$ such that $T(K)$ is isotropic. And $T$ is unique up to rotations
- $L_K = \frac{1}{\sqrt{n}} \mathbb{E}_{\boldsymbol{x}\sim K}[\|\boldsymbol{x}\|_2^2]^{1/2}$ i.e. the average length of points in $K$
- $L_K \geq \Omega(1)$ but the upper bound for $L_K$ is one of the biggest open problems

# Distribution of Mass in Convex Bodies

The following conjectures are equivalent:

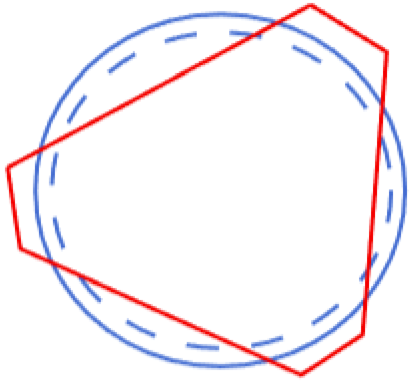
- Isotropic constant conjecture (v1): $L_K \leq \mathcal{O}(1)$
- Isotropic constant conjecture (v2): For any centered convex body $K$ in isotropic position and any $\boldsymbol{y} \in \mathbb{S}^{n-1}$, $\mathbb{E}_{\boldsymbol{x}\sim K}[\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2] \leq \mathcal{O}(1)$
- Slicing conjecture: For any centered convex body $K$ in isotropic position and any $\boldsymbol{\theta} \in \mathbb{S}^{n-1}$, $\mathrm{Vol}_{n-1}(K \cap \boldsymbol{\theta}^\perp) \geq \Omega(1)$
- Bourgain's slicing conjecture: For any centered convex body $K$ with $\mathrm{Vol}_n(K) = 1$, there exists a $\boldsymbol{\theta} \in \mathbb{S}^{n-1}$, $\mathrm{Vol}_{n-1}(K \cap \boldsymbol{\theta}^\perp) \geq \Omega(1)$



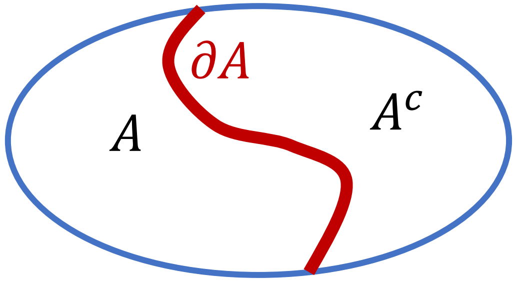
# Distribution of Mass in Convex Bodies

A distribution $\mu$ in $\mathbb{R}^n$ is log-concave if its density is $\mu(\boldsymbol{x}) \propto e^{-f(\boldsymbol{x})}$ for a convex function $f$. Moreover, $\mu$ is isotropic if $\mathbb{E}_\mu[\boldsymbol{x}] = 0$ and $\mathbb{E}_\mu[\boldsymbol{x}\boldsymbol{x}^\top] = \boldsymbol{I}_n$

- Thin-shell conjecture (v1): For any isotropic log-concave distribution $\mu$ in $\mathbb{R}^n$, its thin-shell constant $\sigma_\mu^2 \coloneqq \frac{1}{n}\mathrm{Var}_\mu[\|\boldsymbol{x}\|_2^2] \leq \mathcal{O}(1)$
- Thin-shell conjecture (v2): For any isotropic log-concave distribution $\mu$ in $\mathbb{R}^n$,
$$\int_{\mathbb{R}^n} (\|\boldsymbol{x}\|_2 - \sqrt{n})^2 \mathrm{d}\mu(x) \leq \mathcal{O}(1)$$



- Kannan-Lovász-Simonovits (KLS) conjecture: For any isotropic log-concave distribution $\mu$, there exists a $\psi_\mu > 0$ such that for any $A \subseteq \mathbb{R}^n$
$$\mu^+(\partial A) \geq \psi_\mu^{-1} \cdot \min\{\mu(A), 1 - \mu(A)\}$$
$\psi_\mu \leq \mathcal{O}(1)$ is called the KLS constant



# Distribution of Mass in Convex Bodies

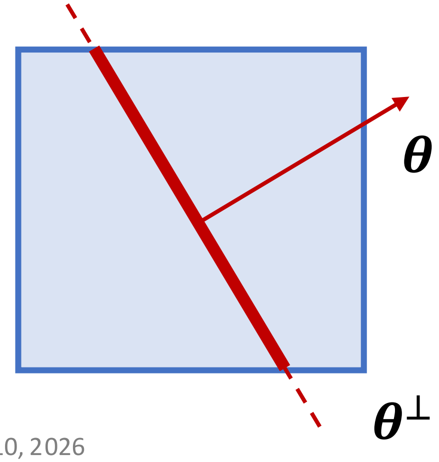
Bourgain's slicing conjecture, thin-shell conjecture, and KLS conjecture are strongly connected

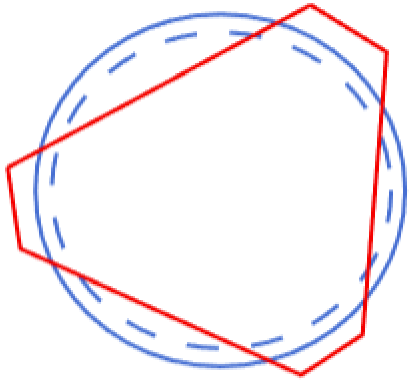
**Recent breakthrough results:**

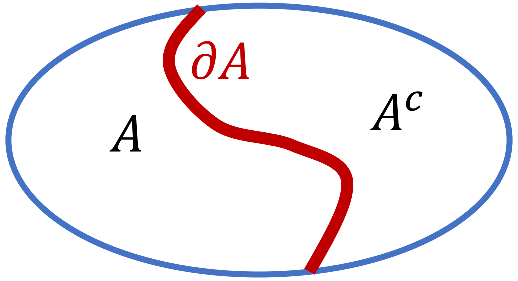
- Klartag '23: $\psi_n \simeq \sqrt{\log n}$
- Klartag-Lehec, Bizeul '25: $L_n = \mathcal{O}(1)$
- Klartag-Lehec '25: $\sigma_n = \mathcal{O}(1)$

Bourgain's slicing theorem

Thin-shell theorem





# Today's Lecture

- Introduction to Asymptotic Convex Geometry
- Why TCS Cares About ACG
  - Dvoretzky's Theorem and JL Lemma
  - John's Theorem and Optimization
  - KLS Conjecture and Geometric Sampling

# Dvoretzky's Theorem and JL Lemma

**Dvoretzky's theorem (Dvoretzky, Milman).**

Let $K \subseteq \mathbb{R}^n$ be a convex symmetric body with $B_2^n \subseteq K$. Let $V \subseteq \mathbb{R}^n$ be a uniformly random subspace of dimension $k \lesssim \frac{\epsilon^2}{\log(1/\epsilon)} \log n$. Then, with high probability,

$$(1-\epsilon)\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_K \leq (1+\epsilon)\|\boldsymbol{x}\|_2 \quad \forall \boldsymbol{x} \in V$$

- Uniformly controls the lengths of uncountably many points

**Lemma (Johnson-Lindenstrauss).**

Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in \mathbb{R}^n$ and $k = \mathcal{O}(\epsilon^{-2} \log m)$. Then, for a Gaussian random matrix $\boldsymbol{A} \sim \mathcal{N}(0,1/k)^{k \times n}$, with high probability,

$$(1-\epsilon)\left\|\boldsymbol{x}_i - \boldsymbol{x}_j\right\|_2 \leq \left\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{A}\boldsymbol{x}_j\right\|_2 \leq (1+\epsilon)\left\|\boldsymbol{x}_i - \boldsymbol{x}_j\right\|_2 \quad \forall i,j \in [m]$$

- Uniformly controls the lengths of finitely many points

# Dvoretzky's Theorem and JL Lemma

- JL lemma is probably the most widely used tool for dimension reduction
- Its proof typically begins by establishing the following weaker result, which then easily implies the full JL lemma (I'll leave thit step to you)

**Lemma (Distributional Johnson-Lindenstrauss).** There exists an $\boldsymbol{A} \sim \mathcal{N}(0,1/m)^{m\times n}$ such that for any $\boldsymbol{x} \in \mathbb{R}^n$, $\|\boldsymbol{Ax}\|_2 \in (1 \pm \epsilon)\|\boldsymbol{x}\|_2$ with probability $1 - 2e^{-\Omega(\epsilon^2 m)}$

*Proof.*

- Let $\boldsymbol{y} := \boldsymbol{Ax} \in \mathbb{R}^m$. Then, for any $i \in [m]$,

$$y_i = \sum_{j=1}^{n} A_{ij}x_j = \sum_{j=1}^{n} \mathcal{N}\left(0, \frac{1}{m}\right) x_j = \sum_{j=1}^{n} \mathcal{N}\left(0, \frac{x_j^2}{m}\right) = \mathcal{N}\left(0, \frac{\|\boldsymbol{x}\|_2^2}{m}\right)$$

- $\mathbb{E}[\|\boldsymbol{y}\|_2^2] = m \cdot (\|\boldsymbol{x}\|_2^2/m) = \|\boldsymbol{x}\|_2^2$

# Dvoretzky's Theorem and JL Lemma

**Lemma (Distributional Johnson-Lindenstrauss).** There exists an $\boldsymbol{A} \sim \mathcal{N}(0,1/m)^{m\times n}$ such that for any $\boldsymbol{x} \in \mathbb{R}^n$, $\|\boldsymbol{Ax}\|_2 \in (1 \pm \epsilon)\|\boldsymbol{x}\|_2$ with probability $1 - 2e^{-\Omega(\epsilon^2 m)}$

*Proof.*

- Let $\boldsymbol{y} \coloneqq \boldsymbol{Ax} \in \mathbb{R}^m$. Then, for any $i \in [m]$, $y_i = \mathcal{N}\left(0, \frac{\|\boldsymbol{x}\|_2^2}{m}\right)$
- $\mathbb{E}[\|\boldsymbol{y}\|_2^2] = m \cdot (\|\boldsymbol{x}\|_2^2/m) = \|\boldsymbol{x}\|_2^2$
- Recall the subgaussian and subgamma random variables in the last lecture:
  - If $X$ is $\sigma^2$-subgaussian, then $X^2$ is $\left(\mathcal{O}(\sigma^4), \mathcal{O}(\sigma^2)\right)$-subgamma
  - $y_i$ is $\|\boldsymbol{x}\|_2^2/m$-subgaussian, so $y_i^2$ is $\left(\mathcal{O}(\|\boldsymbol{x}\|_2^4/m^2), \mathcal{O}(\|\boldsymbol{x}\|_2^2/m)\right)$-subgamma
  - $\|\boldsymbol{y}\|_2^2$ is $\left(\mathcal{O}(\|\boldsymbol{x}\|_2^4/m), \mathcal{O}(\|\boldsymbol{x}\|_2^2/m)\right)$-subgamma

# Dvoretzky's Theorem and JL Lemma

**Lemma (Distributional Johnson-Lindenstrauss).** There exists an $\boldsymbol{A} \sim \mathcal{N}(0,1/m)^{m\times n}$ such that for any $\boldsymbol{x} \in \mathbb{R}^n$, $\|\boldsymbol{A}\boldsymbol{x}\|_2 \in (1 \pm \epsilon)\|\boldsymbol{x}\|_2$ with probability $1 - 2e^{-\Omega(\epsilon^2 m)}$

*Proof.*

- Let $\boldsymbol{y} \coloneqq \boldsymbol{A}\boldsymbol{x} \in \mathbb{R}^m$. Then, for any $i \in [m]$, $y_i = \mathcal{N}\left(0, \frac{\|\boldsymbol{x}\|_2^2}{m}\right)$
- $\mathbb{E}[\|\boldsymbol{y}\|_2^2] = m \cdot (\|\boldsymbol{x}\|_2^2/m) = \|\boldsymbol{x}\|_2^2$
- $\|\boldsymbol{y}\|_2^2$ is $\left(\mathcal{O}(\|\boldsymbol{x}\|_2^4/m), \mathcal{O}(\|\boldsymbol{x}\|_2^2/m)\right)$-subgamma, which implies that

$$\begin{aligned}\Pr\left[\left|\|y\|_2^2 - \|\boldsymbol{x}\|_2^2\right| \geq \epsilon\|\boldsymbol{x}\|_2^2\right] &\leq 2\max\left\{\exp\left(-\frac{\epsilon^2\|\boldsymbol{x}\|_2^4}{\mathcal{O}(\|\boldsymbol{x}\|_2^4/m)}\right), \exp\left(-\frac{\epsilon\|\boldsymbol{x}\|_2^2}{\mathcal{O}(\|\boldsymbol{x}\|_2^2/m)}\right)\right\} \\ &= 2\exp\left(-\Omega(\epsilon^2 m)\right)\end{aligned}$$

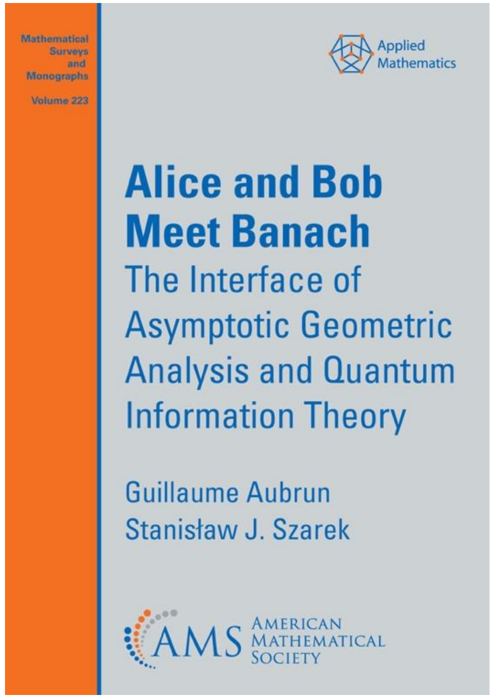
- $\|\boldsymbol{y}\|_2^2 \geq (1+\epsilon)\|\boldsymbol{x}\|^2 \iff \|\boldsymbol{y}\|_2 \geq \sqrt{1+\epsilon}\|\boldsymbol{x}\|_2 = \left(1 + \mathcal{O}(\epsilon)\right)\|\boldsymbol{x}\|_2$

$\blacksquare$

# Dvoretzky's Theorem and Quantum Information

Dvoretzky's Theorem has direct applications in quantum information theory

- Aubrun-Szarek-Werner '10: they used Dvoretzky's Theorem to construct a counterexample that refutes the additivity conjecture for minimal quantum channel output entropy
- Aubrun-Szarek '16: they used Dvoretzky's Theorem to study the complexity of detecting quantum entanglement



# John's Theorem and Optimization

**John's theorem.**

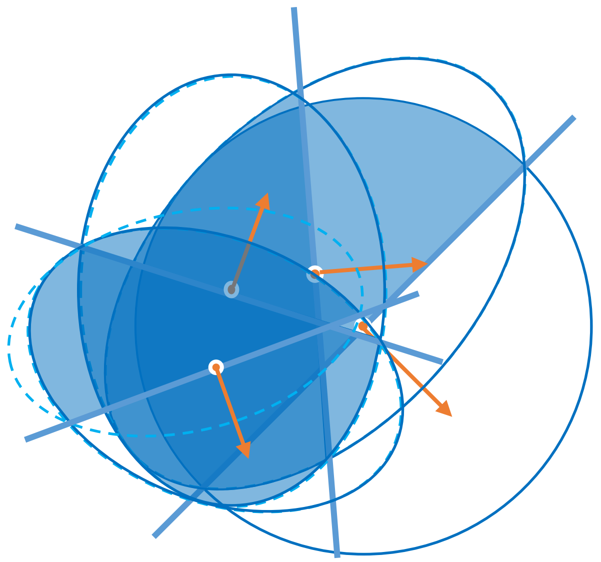
Let $K$ be a convex body. Then there exists a unique maximum-volume ellipsoid inside $K$, called the John ellipsoid $J(K)$. If $K$ is symmetric, then $J(K) \subseteq K \subseteq \sqrt{n}J(K)$; otherwise, $J(K) \subseteq K \subseteq nJ(K)$

- Computing the John ellipsoid is a very interesting algorithmic question.
  - Exact computation requires to solve a semi-definite programming (SDP), which is very expensive.
  - More efficient algorithms have been developed with some approximations and for symmetric polytopes, such as (Cohen-Cousins-Lee-Yang '19, Woodruff-Yasuda'24)
- John ellipsoid has applications in optimization, sampling (Gustafson-Narayanan '23), bandits (Hazan-Karnin '16), differential privacy (Nikolov-Talwar-Zhang '13), coresets (Tukan-Wu-Zhou-Braverman-Feldman '22), randomized numerical linear algebra (Woodruff-Yasuda '22), …

# John's Theorem and Optimization

## Cutting plane method

- Invariant: maintain $\boldsymbol{x}^\star \in K^{(t)}$, where $\boldsymbol{x}^\star$ is the optimal solution of the optimization problem
- Pick large enough $K^{(0)}$
- For $t = 0, 1, 2, \ldots$
  - If $K^{(t)}$ is small enough, **return** the best point we queried
  - Pick some $\boldsymbol{x}^{(t)} \in K^{(t)}$
  - Find a halfspace $H_{\boldsymbol{x}^{(t)}}$ containing $\boldsymbol{x}^\star$
  - Pick some $K^{(t+1)} \supset K^{(t)} \cap H_{\boldsymbol{x}^{(t)}}$



The efficiency depends on how quickly the volume shrinks in each iteration

# John's Theorem and Optimization

## Cutting plane method

| Year | $E^{(k)}$ and $x^{(k)}$ | Rate | Cost/iter |
|---|---|---|---|
| 1965 [15, 16] | Center of gravity | $1-\frac{1}{e}$ | $n^n$ |
| 1979 [19, 17, 12] | Center of ellipsoid | $1-\frac{1}{2n}$ | $n^2$ |
| 1988 [11] | Center of John ellipsoid | 0.87 | $n^{2.878}$ |
| 1989 [18] | Volumetric center | 0.9999999 | $n^{2.378}$ |
| 1995 [5] | Analytic center | 0.9999999 | $n^{2.378}$ |
| 2004 [22] | Center of gravity | $1-\frac{1}{e}$ | $n^6$ |
| 2015 [21] | Hybrid center | $1-10^{-27}$ | $n^2$ (amortized) |

Table 2.1: Different Cutting Plane Methods

https://yintat.com/teaching/cse599-winter18/2.pdf

# John's Theorem and Optimization

## Cutting plane method

- John ellipsoid has the following property:

**Theorem (Tarasovab-Khachiyanab-Èrlikh).**

For any convex set $K$, let $\boldsymbol{x}$ be the center of $J(K)$ and let $H$ be any halfspace containing $\boldsymbol{x}$. Then,

$$\mathrm{Vol}_n(J(K) \cap H^c) \leq 0.87\mathrm{Vol}_n(J(K))$$

- If we cut through the center of $J(K)$ and set $H_{x^{(t)}} \coloneqq H^c$ containing $x^\star$
- Then the $K^{(t+1)}$ has at most 87% of the volume of $K^{(t)}$

# John's Theorem and Optimization

## Cutting plane method

- Another way to pick $\boldsymbol{x}^{(t)}$ is to take the center of gravity (or barycenter) of $K^{(t)}$:

$$\boldsymbol{z} := \mathbb{E}_{\boldsymbol{x}\sim K}[\boldsymbol{x}] = \frac{1}{\mathrm{Vol}_n(K)} \int_K \boldsymbol{x} \mathrm{d}\boldsymbol{x}$$

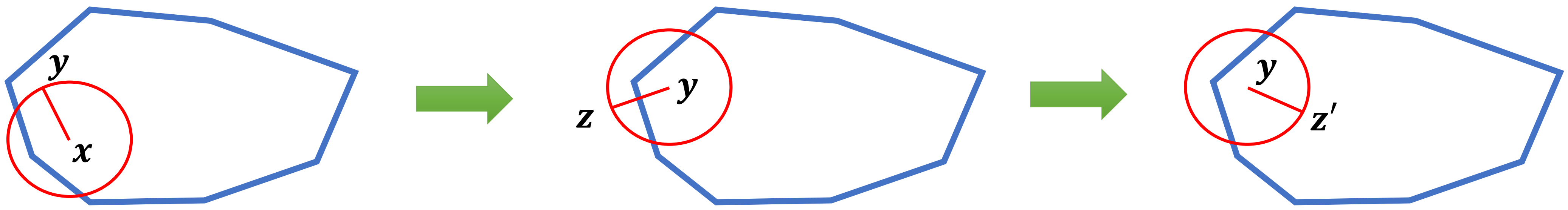
**Lemma (Grünbaum).** Let $K$ be a convex set and $\boldsymbol{z}$ be its barycenter. Let $H$ be a halfspace containing $\boldsymbol{z}$. Then,

$$\mathrm{Vol}_n(K \cap H^c) \leq \left(1 - \frac{1}{e}\right) \mathrm{Vol}_n(K)$$

- $1 - 1/e \approx 0.63 < 0.87$ (the center of $J(K)$)
- However, computing the barycenter is **#P**-hard. But we can approximate it via sampling

# KLS Conjecture and Geometric Sampling

Recall that in the first lecture, we talked about how to estimate the volume of a convex body $K$ by the DFK algorithm and the ball walk:

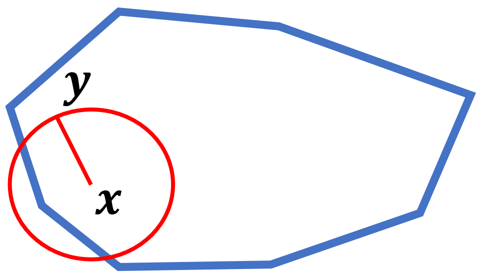


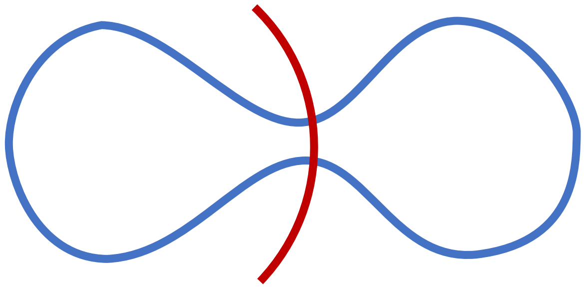

For any distribution $\mu$ in $\mathbb{R}^n$, define its KLS constant as

$$\frac{1}{\psi_\mu} := \inf_{A \subset \mathbb{R}^n} \frac{\mu^+(\partial A)}{\min\{\mu(A), \mu(A^c)\}}$$

**KLS conjecture:** For any log-concave distribution $\mu$ with covariance matrix $\mathbf{\Sigma}$, $\psi_\mu \lesssim \|\mathbf{\Sigma}\|^{1/2}$

# KLS Conjecture and Geometric Sampling



- When $K$ looks like a dumbbell, ball walk may get trapped on one side
- For a set $K$, we can define its Cheeger constant as

$$\phi_K := \inf_{A \subseteq K} \frac{\mathrm{Vol}_{n-1}(\partial A)}{\min\{\mathrm{Vol}_n(A), \mathrm{Vol}_n(K \backslash \mathrm{A})\}}$$

**Theorem** (Kannan-Lovász-Simonovits). For an isotropic convex body $K$, given one random point in $K$, we can generate another in $\mathcal{O}\left(\frac{n^2}{\phi_K^2} \log\left(\frac{1}{\epsilon}\right)\right)$ steps of ball walk with the step size $\delta = \Theta\left(\frac{1}{\sqrt{n}}\right)$

- Since $\phi_K = \psi_\mu^{-1} \geq \psi_n^{-1}$, the number of steps is $\mathcal{O}(n^2 \psi_n^2 \log(1/\epsilon))$
- In the KLS paper, they proved that $\psi_n \leq \sqrt{\mathrm{tr}[\boldsymbol{\Sigma}]} = \sqrt{\mathrm{tr}[\boldsymbol{I}_n]} = \sqrt{n}$, which implies $\tilde{\mathcal{O}}(n^3)$ steps

# Isoperimetric Inequalities and Geometric Sampling

By the current record of the KLS constant $\psi_n \leq \sqrt{\log n}$ (Klartag '23), we have:

**Theorem.** For an isotropic convex body $K$, given one random point in $K$, we can generate another in $\tilde{O}(n^2)$ steps of ball walk

In general, the input convex body $K$ may not be isotropic. We are given $0 < r < R$ such that

$$x_0 + rB_2^n \subset K \subset RB_2^n$$

**Theorem** (Jia-Laddha-Lee-Vempala '21)**.** There is a randomized algorithm that computes an affine transformation $T$ such that $TK$ is nearly isotropic. The query complexity is

$$n^{3.5}\psi_n^2 \operatorname{polylog}(Rn/r, 1/\epsilon) = \tilde{O}(n^{3.5})$$

**Theorem** (Jia-Laddha-Lee-Vempala '21)**.** The query complexity of volume estimation is $\tilde{O}(n^{3.5} + n^3/\epsilon^2)$